

11-COLORED KNOT DIAGRAM WITH FIVE COLORS

TAKUJI NAKAMURA, YASUTAKA NAKANISHI, AND SHIN SATOH

ABSTRACT. We prove that any 11-colorable knot is presented by an 11-colored diagram where exactly five colors of eleven are assigned to the arcs. The number five is the minimum for all non-trivially 11-colored diagrams of the knot. We also prove a similar result for any 11-colorable ribbon 2-knot.

1. INTRODUCTION

The n -colorability introduced by Fox [3] is one of the elementary notion in knot theory, and its properties have been studied in many papers. In 1999, Harary and Kauffman [5] defined a kind of minimal invariant, $C_n(K)$, of an n -colorable knot K . It is essential to consider the case that n is an odd prime; in fact, for composite n , it is reduced to the cases of odd prime factors of n . In this case, we can define a modified version by restricting “effective” n -colorings (cf. [6, 12]).

Let p be an odd prime. A non-trivial p -coloring C of a knot diagram D is regarded as a non-constant map

$$C : \{\text{arcs of } D\} \rightarrow \mathbb{Z}/p\mathbb{Z} = \{0, 1, \dots, p-1\}$$

with a certain condition. For a p -colorable knot K , the number $C_p(K)$ is defined to be the minimum number of $\#\text{Im}(C)$ for all non-trivially p -colored diagrams (D, C) of K . This number has been studied in some papers [2, 4, 7, 8, 10, 11, 13, 15, 17]. In particular, it is shown in [11] that

$$C_p(K) \geq \lfloor \log_2 p \rfloor + 2$$

for any p -colorable knot K , and the equality holds for $p = 3, 5, 7$ [13, 17].

For $p = 11$, we have $C_{11}(K) \geq 5$ by the above inequality or [10, Theorem 2.4]. On the other hand, it is proved in [2] that $C_{11}(K) \leq 6$. If an 11-colored diagram (D, C) satisfies $\#\text{Im}(C) = 5$, then there are two possibilities

$$\text{Im}(C) = \{1, 4, 6, 7, 8\}, \{0, 4, 6, 7, 8\}$$

up to isomorphisms induced by affine maps of $\mathbb{Z}/11\mathbb{Z}$. This split phenomenon is quite different from the cases $p = 3, 5, 7$.

Theorem 1.1. *Any 11-colorable knot K satisfies the following.*

- (i) *There is an 11-colored diagram (D_1, C_1) of K with $\text{Im}(C_1) = \{1, 4, 6, 7, 8\}$.*
- (ii) *There is an 11-colored diagram (D_2, C_2) of K with $\text{Im}(C_2) = \{0, 4, 6, 7, 8\}$.*

We remark that these two sets are *common* 11-minimal sufficient sets of colors but not *universal* ones in the sense of [4]. By Theorem 1.1, we have the following immediately.

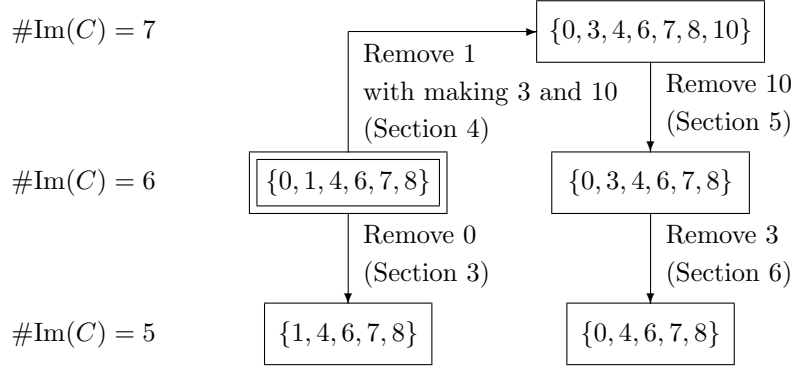
The third author is partially supported by JPSP KAKENHI Grant Number 25400090.

2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 57Q45.

Key words and phrases. knot, diagram, 11-coloring, virtual arc presentation, ribbon 2-knot.

Corollary 1.2. *Any 11-colorable knot K satisfies $C_{11}(K) = 5$.* \square

This paper is organized as follows. In Section 2, we review the palette graph associated with a subset of $\mathbb{Z}/p\mathbb{Z}$ and its fundamental properties. In Section 3, we prove Theorem 1.1(i). The starting point of the proof is a modified version of the theorem in [2]: For any 11-colorable knot K , there is an 11-colored diagram (D, C) of K with $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$. By applying Reidemeister moves to (D, C) suitably, we remove the color 0 from the diagram. Sections 4–6 are devoted to proving Theorem 1.1(ii). We first remove the color 1 from (D, C) as above by allowing the birth of new colors 3 and 10 in Section 4, and then remove the colors 10 and 3 in Sections 5 and 6, respectively. In the last section, we prove a similar result for an 11-colorable ribbon 2-knot.



2. PRELIMINARIES

Throughout this section, p denotes an odd prime.

Definition 2.1. Let S be a subset of $\mathbb{Z}/p\mathbb{Z}$. The *palette graph* $G(S)$ of S is a simple graph such that

- (i) the vertex set of $G(S)$ is S , and
- (ii) two vertices a and $b \in S$ are connected by an edge if and only if $\frac{a+b}{2} \in S$.

By assigning $\frac{a+b}{2}$ to every edge joining a and b , we regard $G(S)$ as a labeled graph. Such an edge is denoted by $\{a | \frac{a+b}{2} | b\}$.

Definition 2.2. For two subsets S and $S' \subset \mathbb{Z}/p\mathbb{Z}$, the palette graphs $G(S)$ and $G(S')$ are said to be *isomorphic* if there is a bijection $f : S \rightarrow S'$ such that $\frac{a+b}{2} \in S$ if and only if $\frac{f(a)+f(b)}{2} \in S'$. We denote it by $G(S) \cong G(S')$.

Lemma 2.3. *If $S \subset S' \subset \mathbb{Z}/p\mathbb{Z}$, then $G(S)$ is a subgraph of $G(S')$, which is obtained from $G(S')$ by deleting the vertices in $S' \setminus S$ and the edges whose labels belong to $S' \setminus S$.*

Proof. This follows from definition immediately. \square

Theorem 2.4 ([11]). *If the palette graph $G(S)$ is connected with $\#S > 1$, then we have $\#S \geq \lfloor \log_2 p \rfloor + 2$.* \square

Lemma 2.5. *Let S be a subset of $\mathbb{Z}/p\mathbb{Z}$ such that $G(S)$ is connected with $\#S = \lfloor \log_2 p \rfloor + 2$. Put $U = \{S' \subset \mathbb{Z}/p\mathbb{Z} \mid G(S') \cong G(S)\}$. Then we have $\#U = p(p-1)$.*

Proof. Let T be a maximal tree of $G(S)$. Let v_1, v_2, \dots, v_k be the vertices of T , and e_1, e_2, \dots, e_{k-1} the edges of T , where $k = \#S = \lfloor \log_2 p \rfloor + 2$. Let $A = (a_{ij})$ be the $(k-1) \times k$ matrix with \mathbb{Z} -entries defined by

$$a_{ij} = \begin{cases} 1 & (e_i \text{ is incident to } v_j), \\ -2 & (\text{the label of } e_i \text{ is } v_j), \\ 0 & (\text{otherwise}). \end{cases}$$

Let A' be the $(k-1) \times (k-1)$ matrix obtained from A by deleting the k th column. It is known in [11] that

- (i) $\det(A')$ is odd,
- (ii) $|\det(A')| < 2^{k-1}$, and
- (iii) $\det(A')$ is divisible by p .

Since $2^{k-2} < p \leq |\det(A')| < 2^{k-1}$, we have $|\det(A')| = p$. This implies that the corank of A with $\mathbb{Z}/p\mathbb{Z}$ -entries is exactly equal to 2.

Let $V = \{\mathbf{x} \mid A\mathbf{x} \equiv \mathbf{0} \pmod{p}\}$ denote the solution space. By the above argument, we have

$$V = \{\lambda \cdot {}^t(v_1, v_2, \dots, v_k) + \mu \cdot {}^t(1, 1, \dots, 1) \mid \lambda, \mu \in \mathbb{Z}/p\mathbb{Z}\}.$$

Since the elements of U are identified with the vectors of V whose entries are all distinct. Such a vector is obtained by the condition $\lambda \not\equiv 0 \pmod{p}$. Therefore, we have $\#U = p(p-1)$. \square

Theorem 2.6. *Let S and S' be subsets of $\mathbb{Z}/p\mathbb{Z}$. Suppose that $G(S)$ and $G(S')$ are connected with $\#S = \#S' = \lfloor \log_2 p \rfloor + 2$. Then the following are equivalent.*

- (i) *The palette graphs $G(S)$ and $G(S')$ are isomorphic.*
- (ii) *There exist $\alpha \not\equiv 0$ and $\beta \in \mathbb{Z}/p\mathbb{Z}$ such that the affine map $f(x) = \alpha x + \beta$ satisfies $f(S) = S'$.*

Proof. (ii) \Rightarrow (i). Since $\alpha \not\equiv 0 \pmod{p}$, $f : S \rightarrow S'$ is a bijection. Furthermore, $\frac{a+b}{2} \in S$ holds if and only if $f(\frac{a+b}{2}) = \frac{f(a)+f(b)}{2} \in f(S) = S'$ holds.

(i) \Rightarrow (ii). By the above argument, we have

$$U \supset \{f(S) \mid f(x) = \alpha x + \beta, \alpha \not\equiv 0, \beta \in \mathbb{Z}/p\mathbb{Z}\},$$

where U is the set in Lemma 2.5. Since these two sets have the same number of elements by Lemma 2.5, they are the same set. \square

Let D be a diagram of a knot K . We regard D as a disjoint union of arcs whose endpoints are under-crossings. Fox [3] introduced the notion of p -colorings: A map $C : \{\text{arcs of } D\} \rightarrow \mathbb{Z}/p\mathbb{Z}$ is a p -coloring if $a+b \equiv 2c \pmod{p}$ holds at every crossing, where a and b are the elements assigned to the under-arcs by C , and c is the one to the over-arc. The triple $\{a|c|b\}$ is called the *color* of the crossing. The assigned element of an arc of D is called the *color* of the arc. If the color of an arc is a , then the arc is called an *a-arc*.

In a p -colored diagram (D, C) , the crossing of color $\{a|a|a\}$ is called *trivial*, and otherwise *non-trivial*. If C is a constant map, it is called a *trivial p -coloring*, and otherwise, *non-trivial*. In other words, a p -coloring C is non-trivial if and only if $\#\text{Im}(C) > 1$. If a knot K admits a non-trivially p -colored diagram (D, C) , K is called *p -colorable*.

For a p -colorable knot K , we denote by $C_p(K)$ the minimum number of $\#\text{Im}(C)$ for all non-trivially p -colored diagram (D, C) of K [5]. For the study of this number, it is helpful to use the palette graph $G(\text{Im}(C))$ of the image $\text{Im}(C) \subset \mathbb{Z}/p\mathbb{Z}$ in the following sense.

Lemma 2.7. *If $\{a|c|b\}$ is a non-trivial color of a crossing of a p -colored diagram (D, C) , then the palette graph $G(\text{Im}(C))$ has an edge $\{a|c|b\}$.*

Proof. Since $a + b \equiv 2c \pmod{p}$ holds, the lemma follows by definition. \square

Lemma 2.8. *The palette graph $G(\text{Im}(C))$ of a p -colored diagram (D, C) of a knot is connected.*

Proof. Let a and b be vertices of $G(\text{Im}(C))$. By definition, we have an a -arc and a b -arc of D . Since D is a diagram of a knot (not a link), we can walk along D from the a -arc to the b -arc. Let $\{a_i|c_i|a_{i+1}\}$ ($1 \leq i \leq k-1$) be the colors of non-trivial under-crossings on the path such that $a_1 = a$ and $a_k = b$. Then the vertices a and b in the palette graph are connected by a sequence of edges $\{a_i|c_i|a_{i+1}\}$ ($1 \leq i \leq k-1$). \square

Theorem 2.9 ([11]). *Any non-trivial p -colored diagram (D, C) of a knot satisfies $\#\text{Im}(C) \geq \lfloor \log_2 p \rfloor + 2$. Therefore, we have $C_p(K) \geq \lfloor \log_2 p \rfloor + 2$ for any p -colorable knot K .*

Proof. This follows from Theorem 2.4 and Lemma 2.8. \square

Lemma 2.10. *Let (D, C) be a non-trivially p -colored diagram of a knot K , and $f : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ an affine map defined by $f(x) = \alpha x + \beta$ with $\alpha \not\equiv 0$ and $\beta \in \mathbb{Z}/p\mathbb{Z}$. Then there is a non-trivially p -colored diagram (D, C') of K such that $\text{Im}(C') = f(\text{Im}(C))$.*

Proof. It is easy to see that the composition $C' = f \circ C$ is also a non-trivial p -coloring of D . \square

Now, we consider the case $p = 11$. By Theorem 2.4, if the palette graph $G(S)$ of a subset $S \subset \mathbb{Z}/11\mathbb{Z}$ is connected with $\#S > 1$, then $\#S \geq 5$.

Theorem 2.11 ([4, Theorem 12]). *Let S be a subset of $\mathbb{Z}/11\mathbb{Z}$. If the palette graph $G(S)$ is connected with $\#S = 5$, then $G(S)$ is isomorphic to $G(\{1, 4, 6, 7, 8\})$ or $G(\{0, 4, 6, 7, 8\})$ as shown in Figure 1.* \square

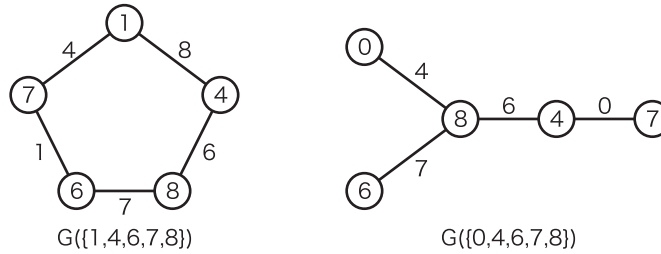


FIGURE 1.

By Theorem 2.9 or [10, Theorem 2.4], we have $C_{11}(K) \geq 5$. The following theorem implies that $C_{11}(K) = 5$ or 6.

Theorem 2.12 ([2]). *For any 11-colorable knot K , there is a non-trivially 11-colored diagram (D, C) of K with $\text{Im}(C) \subset \{0, 1, 4, 6, 7, 8\}$.* \square

Figure 2 shows the palette graph $G(\{0, 1, 4, 6, 7, 8\})$. By Lemma 2.3, the two graphs in Theorem 2.11 are obtained from this graph by deleting the vertex a and the edges labeled a for $a = 0, 1$, respectively.

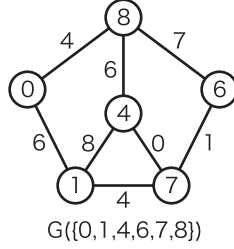


FIGURE 2.

It is useful for our argument to modify Theorem 2.12 slightly as follows.

Lemma 2.13. *For any 11-colorable knot K , there is an 11-colored diagram (D, C) of K with $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$.*

Proof. We may assume that (D, C) satisfies Theorem 2.12; that is, it is a non-trivially 11-colored diagram with $\text{Im}(C) \subset \{0, 1, 4, 6, 7, 8\}$. We remark that $\#\text{Im}(C) \geq 5$ by Theorem 2.9.

$4, 6, 7 \in \text{Im}(C)$. Assume that $4 \notin \text{Im}(C)$. It follows that $\text{Im}(C) = \{0, 1, 6, 7, 8\}$. The palette graph $G(\text{Im}(C))$ is as shown in the left of Figure 3 by Lemma 2.3, which contradicts to the connectivity in Lemma 2.8. We can also prove $6, 7 \in \text{Im}(C)$ by a similar argument. See the center and right of the figure.

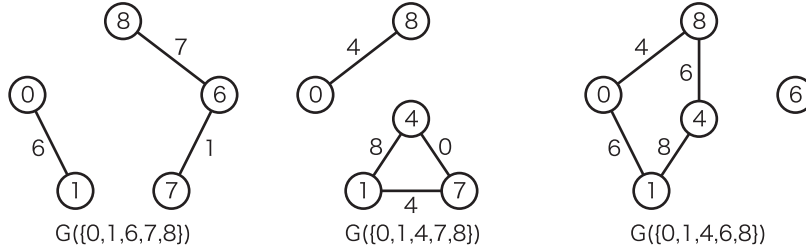


FIGURE 3.

$0 \in \text{Im}(C)$. Assume that $0 \notin \text{Im}(C)$. It follows that $\text{Im}(C) = \{1, 4, 6, 7, 8\}$ and its palette graph is as shown in the left of Figure 1. Then we see that (D, C) has a crossing of color $\{6|1|7\}$ or $\{1|8|4\}$. In fact, if we delete the corresponding edges both, the resulting graph becomes disconnected. By deforming the diagram near these crossings as shown in Figure 4, we can produce a 0-arc. We replace the original diagram with the new one as (D, C) .

$1 \in \text{Im}(C)$. Assume that $1 \notin \text{Im}(C)$. Then we have $\text{Im}(C) = \{0, 4, 6, 7, 8\}$ and its palette graph is as shown the right of Figure 1. Since (D, C) must have a crossing

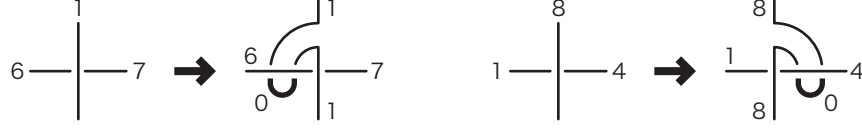


FIGURE 4.

of color $\{0|4|8\}$ by a similar reason to the above case, we deform the diagram near the crossing to make a 1-arc. See Figure 5.

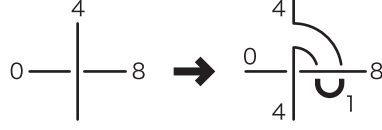


FIGURE 5.

$8 \in \text{Im}(C)$. Assume that $8 \notin \text{Im}(C)$. Then we have $\text{Im}(C) = \{0, 1, 4, 6, 7\}$ and its palette graph is as shown in the left of Figure 6. We remark that the map $f : \mathbb{Z}/11\mathbb{Z} \rightarrow \mathbb{Z}/11\mathbb{Z}$ defined by $f(x) = 7x + 6$ induces the isomorphism between $G(\{0, 4, 6, 7, 8\})$ and $G(\{0, 1, 4, 6, 7\})$. The existence of such a map is guaranteed by Theorem 2.6. Since (D, C) has a crossing of color $\{4|0|7\}$, we deform the diagram near the crossing as shown in the right of the figure so that we obtain an 8-arc. \square

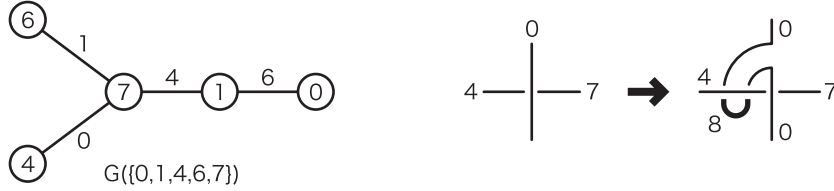


FIGURE 6.

3. PROOF OF THEOREM 1.1(i)

Lemma 3.1. *For any 11-colorable knot K , there is an 11-colored diagram (D, C) of K such that*

- (i) $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$, and
- (ii) *there is no crossing of color $\{*\mid 0\mid *\}$.*

Proof. We may assume that (D, C) satisfies Lemma 2.13. There are two types of crossings of (D, C) whose over-arc is a 0-arc; that is, $\{0|0|0\}$ and $\{4|0|7\}$. In fact, in the palette graph $G(\{0, 1, 4, 6, 7, 8\})$, the only edge labeled 0 connects 4 and 7.

First, we assume that (D, C) has crossings of color $\{4|0|7\}$. By deforming the diagram near the crossings as shown in Figure 7, we can eliminate all the crossings

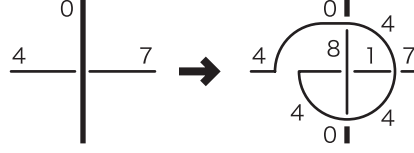


FIGURE 7.

of color $\{4|0|7\}$. We remark that the set of colors which are appeared in the diagram does not change.

Next, we assume that (D, C) has a crossing of color $\{0|0|0\}$, say x . Walking along the diagram from x , let y be the non-trivial crossing which we meet first. If there are crossings of color $\{0|0|0\}$ between x and y , we replace the original x with the nearest one to y . Therefore, we may assume that there is no crossing between x and y .

There are two cases with respect to the color of y . In fact, in the palette graph $G(\{0, 1, 4, 6, 7, 8\})$, there are two edges incident to the vertex 0, which implies that the color of y is $\{0|6|1\}$ or $\{0|4|8\}$. In each case, we deform the diagram (D, C) near x and y as shown in Figure 8, so that the number of crossings of $\{0|0|0\}$ is decreased. By repeating this process, we obtain a diagram with no crossing of $\{0|0|0\}$ finally. \square

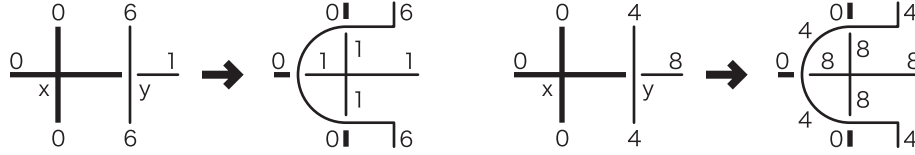


FIGURE 8.

Proof of Theorem 1.1(i). We may assume that (D, C) satisfies Lemma 3.1. If there is a 0-arc, it is not an over-arc of any crossing, and its endpoints are the under-crossings of color $\{0|4|8\}$ or $\{0|6|1\}$. In fact, there are two edges incident to the vertex 0 in $G(\{1, 4, 6, 7, 8\})$. We have three cases with respect to the colors of the crossings of the endpoints of a 0-arc;

- (i) $\{0|4|8\}$ and $\{0|6|1\}$,
- (ii) $\{0|4|8\}$ both, and
- (iii) $\{0|6|1\}$ both.

For the case (i), we deform the 6-arc over the crossing of $\{0|4|8\}$ to eliminate the 0-arc. See the top of Figure 9. For the case (ii), we deform one of the crossings of color $\{0|4|8\}$ as shown in the figure so that we reduce this case to (i). Similarly, for the case (iii), we deform one of the crossings of color $\{0|6|1\}$ as shown in the figure so that we reduce this case to (i). See the bottom of the figure. \square

Corollary 3.2. *For any 11-colorable knot K and $a \neq b \in \mathbb{Z}/11\mathbb{Z}$, there is an 11-colored diagram (D, C) of K with*

$$\text{Im}(C) = \{a, b, 3a + 9b, 6a + 6b, 10a + 2b\}.$$

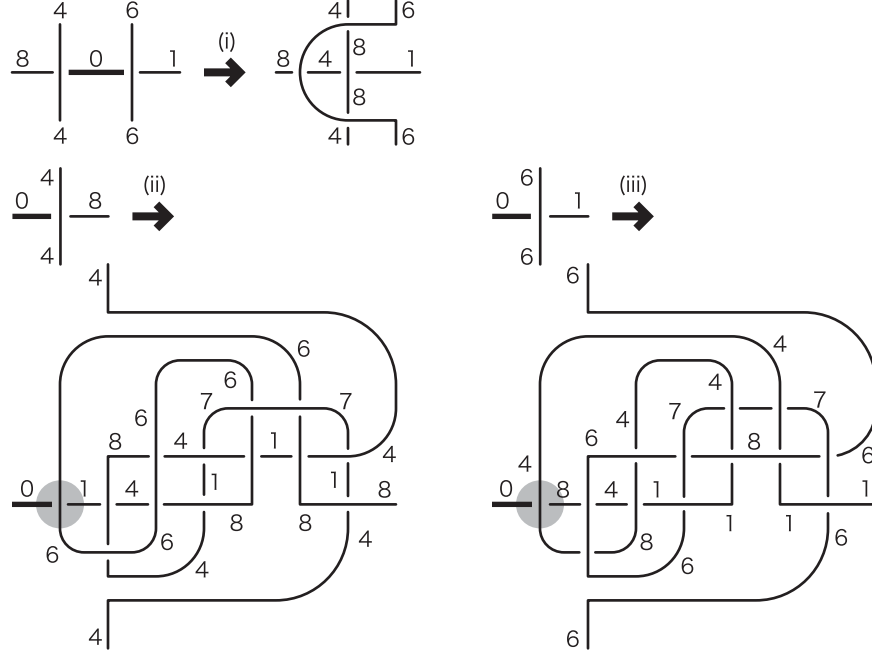


FIGURE 9.

Proof. Let $f : \mathbb{Z}/11\mathbb{Z} \rightarrow \mathbb{Z}/11\mathbb{Z}$ be the affine map defined by $f(x) = 4(b-a)(x-1) + a$. Since the map f satisfies

$$f(1) = a, \quad f(4) = b, \quad f(6) = 3a + 9b, \quad f(7) = 10a + 2b, \quad f(8) = 6a + 6b,$$

we have the conclusion by Lemma 2.10 and Theorem 1.1(i). \square

4. PROOF OF THEOREM 1.1(ii)–PART I

Lemma 4.1. *For any 11-colorable knot K , there is an 11-colored diagram (D, C) of K such that*

- (i) $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$, and
- (ii) *there is no crossing of color $\{6|6|6\}$.*

Proof. We may assume that (D, C) satisfies Lemma 2.13. Assume that (D, C) has a crossing of color $\{6|6|6\}$, say x . Walking along the diagram from x , let y be the first non-trivial *under*-crossing. If there are crossings of color $\{6|6|6\}$ between x and y , then we replace the original x with the nearest one to y . Then we have the following:

- (i) There is no crossing of $\{6|6|6\}$ between x and y by assumption.
- (ii) Every crossing between x and y is of color $\{0|6|1\}$ or $\{4|6|8\}$; for there are exactly two edges labeled 6 in the palette graph $G(\{0, 1, 4, 6, 7, 8\})$.
- (iii) The color of y is $\{6|1|7\}$ or $\{6|7|8\}$; for there are exactly two edges incident to the vertex 6 in the palette graph, which are labeled 1 and 7, respectively.

Assume that there are crossings between x and y . Let z be the nearest crossing to x among them. We deform the diagram near x and z as shown in the upper row

of Figure 10, so that the number of crossings between x and y is decreased. By repeating this process, we may assume that there is no crossing between x and y . Then we deform the diagram near x and y as shown in the lower row of the figure to eliminate the color $\{6|6|6\}$. By repeating this process, we obtain a diagram with no crossing of $\{6|6|6\}$ finally. \square

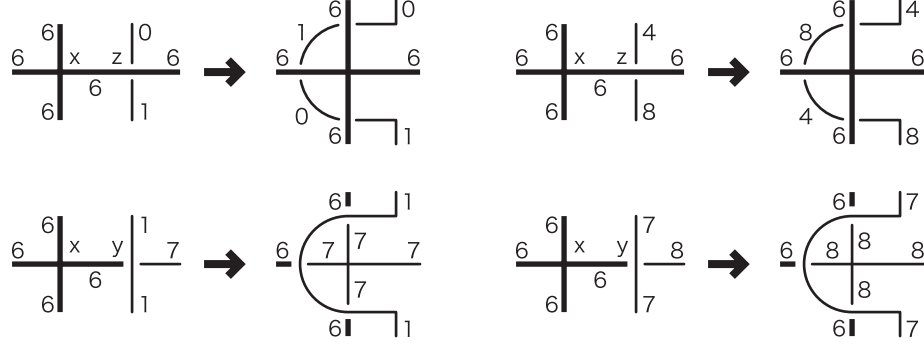


FIGURE 10.

Lemma 4.2. *For any 11-colorable knot K , there is an 11-colored diagram (D, C) of K such that*

- (i) $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$, and
- (ii) *there is no crossing of color $\{1|1|1\}$ or $\{6|6|6\}$.*

Proof. We may assume that (D, C) satisfies Lemma 4.1. Assume that (D, C) has a crossing of color $\{1|1|1\}$, say x . Walking along the diagram from x , let y be the first non-trivial crossing. If there are crossing of $\{1|1|1\}$ between x and y , then we replace the original x with the nearest one to y .

In the palette graph $G(\{0, 1, 4, 6, 7, 8\})$, there are exactly three edges incident to the vertex 1 whose labels are 4, 6, and 8, and there is only one edge whose label is 1. Therefore, the color of the crossing y is $\{1|4|7\}$, $\{1|6|0\}$, $\{1|8|4\}$, or $\{6|1|7\}$.

We deform the diagram near x and y as shown in Figure 11 so that the number of crossings of color $\{1|1|1\}$ is decreased. By repeating this process, we obtain a diagram with no crossing of $\{1|1|1\}$. \square

Lemma 4.3. *For any 11-colorable knot K , there is a non-trivially 11-colored diagram (D, C) of K such that*

- (i) $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$, and
- (ii) *there is no crossing of color $\{*\!|1|\!*\}$ or $\{6|6|6\}$.*

Proof. We may assume that (D, C) satisfies Lemma 4.2. Assume that (D, C) has a crossing of color $\{*\!|1|\!*\}$. Since there is only one edge labeled 1 in the palette graph $G(\{0, 1, 4, 6, 7, 8\})$, the color of the corresponding crossing is $\{6|1|7\}$.

There is a 4-arc in (D, C) . We will pull the 4-arc toward each crossing of $\{6|1|7\}$. In the process, we can assume that the 4-arc crosses over several arcs whose colors are 0, 1, 4, 6, 7, 8 missing 6. In fact, since there is no crossing of $\{6|6|6\}$, the set of 6-arcs is a disjoint union of intervals in the plane, and the complement in the plane

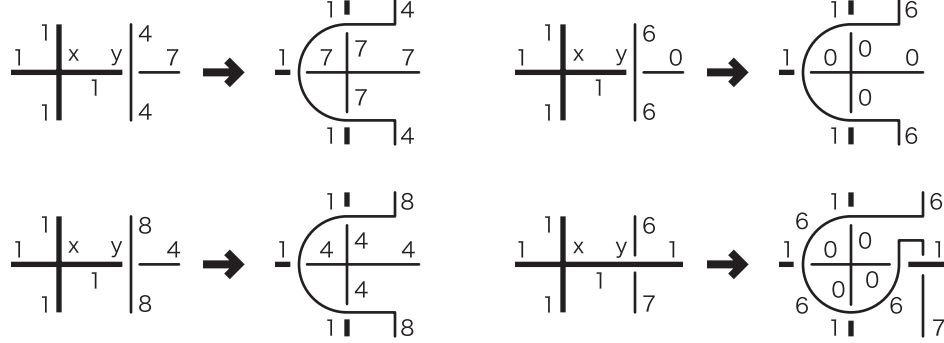


FIGURE 11.

is connected. When the 4-arc crosses over an a -arc for $a = 0, 1, 4, 7, 8$, we have a pair of new crossings of color

$$\{a|4|8-a\} = \{0|4|8\}, \{1|4|7\}, \{4|4|4\}, \{7|4|1\}, \{8|4|0\},$$

respectively. See the left of Figure 12. We remark that any vertex of the palette graph $G(\{0, 1, 4, 6, 7, 8\})$ other than 6 is 4 itself or incident to an edge labeled 4.

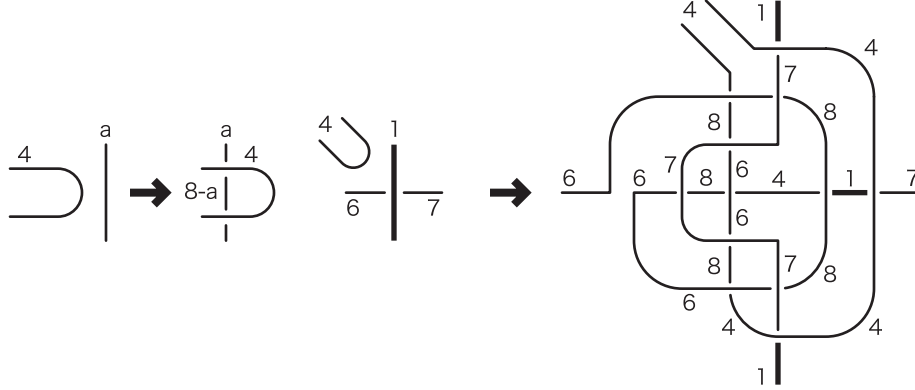


FIGURE 12.

By deforming the diagram near every crossing of $\{6|1|7\}$ with a 4-arc as shown in the right of the figure, we obtain a diagram with no crossing of $\{6|1|7\}$. Then the arcs in the obtained diagram are colored by 0, 1, 4, 6, 7, 8 and there is no crossing of $\{*|1|*\}$ or $\{6|6|6\}$. \square

Lemma 4.4. *For any 11-colorable knot K , there is an 11-colored diagram (D, C) of K such that*

- (i) $\text{Im}(C) = \{0, 3, 4, 6, 7, 8, 10\}$,
- (ii) *there is no crossing of color $\{6|6|6\}$, and*
- (iii) *if $\{a|b|c\}$ is the color of a crossing and at least one of a, b, c is 3 or 10, then it is one of*

$$\{0|3|6\}, \{0|7|3\}, \{3|0|8\}, \{4|7|10\}, \{7|3|10\}, \{3|3|3\}.$$

Proof. We may assume that (D, C) satisfies Lemma 4.3. Since there are three edges incident to the vertex 1 in the palette graph $G(\{0, 1, 4, 6, 7, 8\})$, every crossing with a 1-arc is of color $\{1|4|7\}$, $\{1|6|0\}$, or $\{1|8|4\}$. If there is a crossing of $\{1|8|4\}$, we deform the 4-arc near the crossing as shown in Figure 13 to replace the crossing with the one of color $\{1|4|7\}$. Therefore, we may assume that there is no crossing of $\{1|8|4\}$.

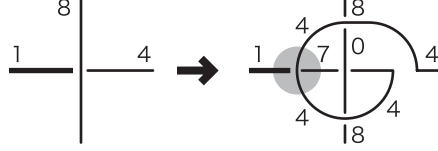


FIGURE 13.

There is a 0-arc in (D, C) . We will pull the 0-arc toward each crossing of $\{1|4|7\}$. In the process, we can assume that the 0-arc crosses over several a -arcs for $a \in \{0, 1, 4, 7, 8\}$ missing 6 by the same reason in the proof of Lemma 4.3; that is, there is no crossing of $\{6|6|6\}$. When the 0-arc crosses over an a -arc, we have a pair of new crossings of color

$$\{a|0|-a\} = \{0|0|0\}, \{1|0|10\}, \{4|0|7\}, \{7|0|4\}, \{8|0|3\},$$

respectively. We remark that the new colors 3 and 10 appear at the crossings of $\{1|0|10\}$ and $\{3|0|8\}$. See Figure 14.



FIGURE 14.

By deforming the diagram near every crossing of $\{1|4|7\}$ with a 0-arc as shown in Figure 15, we remove all the crossings of $\{1|4|7\}$ and produce the color 10 at the crossings of $\{1|0|10\}$ and $\{4|7|10\}$.

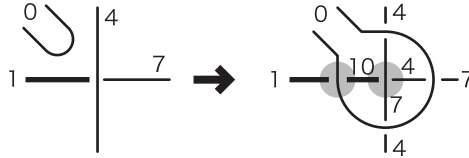


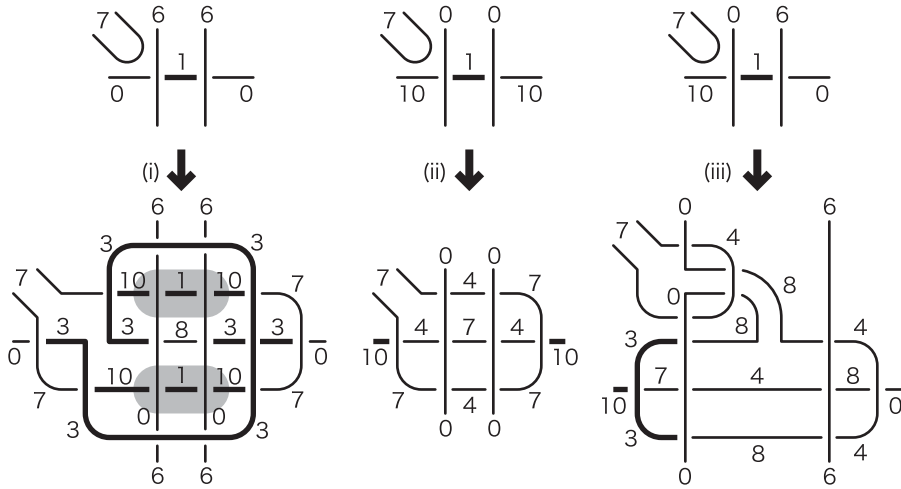
FIGURE 15.

There is a 7-arc in (D, C) . We will pull the 7-arc toward each 0-arc. In the process, we can assume that the 7-arc crosses over several a -arcs for $a \in$

$$\{a|7|3 - a\} = \{0|7|3\}, \{3|7|0\}, \{4|7|10\}, \{6|7|8\}, \{7|7|7\}, \{8|7|6\}, \{10|7|4\},$$

Now, every crossing with a 1-arc is of color $\{1|6|0\}$ or $\{1|0|10\}$. The endpoints every 1-arc are under-crossings of color

- For every 1-arc of type (i), we deform the diagram near the 1-arc equipped with a 7-arc into type (ii) as shown in the left of Figure 16. Here, the colors 3 and 10 appear at the crossings of $\{0|7|3\}$, $\{7|3|10\}$, $\{0|3|6\}$, $\{3|3|3\}$, and $\{3|0|8\}$.



For every 1-arc of type (ii) or (iii), we deform the diagram near the 1-arc with a 7-arc as shown in the center and right of the figure, so that we can remove all the 1-arcs from the diagram. We remark that the colors 3 and 10 appear at the crossings of $\{4|7|10\}$ for (ii) and $\{3|0|8\}$ and $\{7|3|10\}$ for (iii).

We remark that the 11-colored diagram (D, C) in Lemma 4.4 has no crossing of color $\{3|10|6\}$, $\{6|8|10\}$, or $\{10|10|10\}$. In particular, there is no crossing whose over-arc is colored 10.

5. PROOF OF THEOREM 1.1(ii)–PART II

Let G_1 be the graph obtained from the palette graph $G(\{0, 4, 6, 7, 8\})$ by adding two vertices 3 and 10 and five edges

$$\{0|3|6\}, \{0|7|3\}, \{3|0|8\}, \{4|7|10\}, \{7|3|10\}.$$

See Figure 17. In other words, G_1 is obtained from $G(\{0, 3, 4, 5, 6, 8, 10\})$ by deleting the edges $\{3|10|6\}$ and $\{6|8|10\}$.

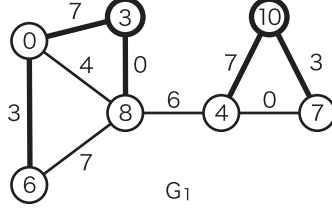


FIGURE 17.

Assume that (D, C) satisfies Lemma 4.4. If $\{a|c|b\}$ is the non-trivial color of a crossing of (D, C) , then the palette graph G_1 has the corresponding edge $\{a|c|b\}$.

Lemma 5.1. *For any 11-colorable knot K , there is an 11-colored diagram (D, C) of K such that*

- (i) $\text{Im}(C) = \{0, 3, 4, 6, 7, 8\}$, and
- (ii) *there is no crossing of color $\{6|6|6\}$.*

Proof. We may assume that (D, C) satisfies Lemma 4.4. Since the graph G_1 has no edge whose label is 10 and (D, C) has no crossing of $\{10|10|10\}$, we see that there is no crossing of color $\{*\mid 10\mid *\}$.

Since there are two edges incident to the vertex 10 in G_1 , every crossing with a 10-arc is of color $\{4|7|10\}$ or $\{7|3|10\}$. If there is a crossing of $\{7|3|10\}$, we deform the 7-arc near the crossing as shown in the left of Figure 18 to replace the crossing with one of $\{4|7|10\}$. We remark that the crossings of $\{0|7|3\}$ and $\{4|0|7\}$ are also produced. Therefore, we may assume that there is no crossing of $\{7|3|10\}$.

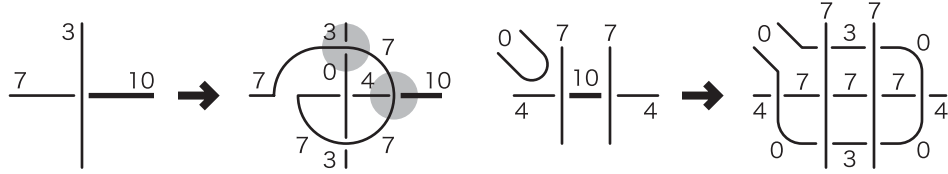


FIGURE 18.

There is a 0-arc in (D, C) . We will pull the 0-arc toward each 10-arc. In the process, we can assume that the 0-arc crosses over several arcs whose colors are 0, 3, 4, 7, 8 missing 6 and 10. In fact, since there is no crossing of color

$$\{3|10|6\}, \{2|6|10\}, \{6|8|10\}, \{6|6|6\}, \{10|10|10\},$$

the set of 6- and 10-arcs is a disjoint union of intervals, and the complement in the plane is connected. When the 0-arc crosses over an a -arc for $a = 0, 3, 4, 7, 8$, we have a pair of new crossings of color

$$\{a|0| - a\} = \{0|0|0\}, \{3|0|8\}, \{4|0|7\}, \{7|0|4\}, \{8|0|3\},$$

respectively. We remark that any vertex of G_1 other than 6 and 10 is 0 itself or incident to an edge labeled 0.

We deform the diagram near every 10-arc with a 0-arc as shown in the right of the figure, so that we remove all the 10-arcs from the diagram. We remark that the crossings of $\{0|7|3\}$, $\{4|0|7\}$, and $\{7|7|7\}$ are produced. \square

6. PROOF OF THEOREM 1.1(ii)–PART III

Lemma 6.1. *For any 11-colorable knot K , there is an 11-colored diagram (D, C) of K such that*

- (i) $\text{Im}(C) = \{0, 3, 4, 6, 7, 8\}$,
- (ii) *there is no crossing of color $\{3|3|3\}$, $\{4|4|4\}$, or $\{6|6|6\}$.*

Proof. We may assume that (D, C) satisfies Lemma 5.1 with $\text{Im}(C) = \{0, 3, 4, 6, 7, 8\}$. Figure 19 shows the palette graph $G(\{0, 3, 4, 6, 7, 8\})$, which is obtained from G_1 by deleting the vertex 10 and its incident edges $\{4|7|10\}$ and $\{7|3|10\}$.

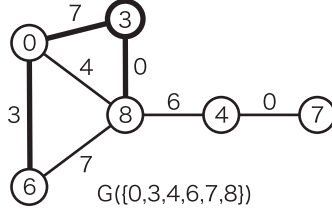


FIGURE 19.

There is a 0-arc in (D, C) . Similarly to the proof of Lemma 5.1, we can pull the 0-arc freely without producing new colors. We remark that any vertex of $G(\{0, 3, 4, 6, 7, 8\})$ other than 6 is 0 itself or incident to an edge labeled 0. Then we deform the diagram near every 3- or 4-arc with a 0-arc as shown in Figure 20 so that there is no crossing of $\{3|3|3\}$ or $\{4|4|4\}$. \square

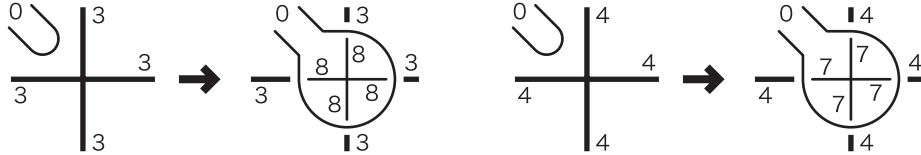


FIGURE 20.

Lemma 6.2. *For any 11-colorable knot K , there is an 11-colored diagram (D, C) of K such that*

- (i) $\text{Im}(C) = \{0, 3, 4, 6, 7, 8\}$,

(ii) *there is no crossing of color $\{*\mid 3\}$, $\{4\mid 4\}$, or $\{6\mid 6\}$.*

Proof. We may assume that (D, C) satisfies Lemma 6.1. In the palette graph $G(\{0, 3, 4, 6, 7, 8\})$, there is only one edge whose label is 3. Therefore, every crossing whose over-arc is 3 has the color $\{0\mid 3\}$.

There is a 7-arc in (D, C) . We will pull the 7-arc toward each crossing of $\{0\mid 3\}$. Since there is no crossing of $\{4\mid 4\}$, we can assume that the 7-arc crosses over several arcs whose colors are 0, 3, 6, 7, 8 missing 4. If the 7-arc crosses an a -arc for $a \in \{0, 3, 6, 7, 8\}$, then we have a pair of new crossings of color

$$\{a\mid 7\mid 3 - a\} = \{0\mid 7\mid 3\}, \{3\mid 7\mid 0\}, \{6\mid 7\mid 8\}, \{7\mid 7\mid 7\}, \{8\mid 7\mid 6\},$$

respectively. We remark that any vertex of $G(\{0, 3, 4, 6, 7, 8\})$ other than 4 is 7 itself or incident to an edge labeled 7. We deform the diagram near every crossing of $\{0\mid 3\}$ equipped with a 7-arc as shown in Figure 21 to remove all the crossings of $\{0\mid 3\}$. \square

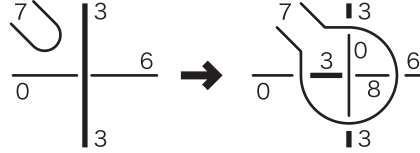


FIGURE 21.

Proof of Theorem 1.1(ii). We may assume that (D, C) satisfies Lemma 6.2. Since there are two edges incident to the vertex 3 in $G(\{0, 3, 4, 6, 7, 8\})$, every crossing with a 3-arc is of color $\{3\mid 0\mid 8\}$ or $\{0\mid 7\mid 3\}$. Therefore, the endpoints of every 3-arc are under-crossings of color

- (i) $\{3\mid 0\mid 8\}$ and $\{0\mid 7\mid 3\}$,
- (ii) $\{3\mid 0\mid 8\}$ both, or
- (iii) $\{0\mid 7\mid 3\}$ both.

For every 3-arc of type (i), we deform the diagram near the crossing of $\{0\mid 7\mid 3\}$, which reduces a 3-arc of type (ii). See the left of Figure 22. Therefore, we may assume that there is no 1-arc of type (i).

To remove a 3-arc of type (ii), We will pull a 7-arc toward the 3-arc. Since there is no crossing of $\{4\mid 4\mid 4\}$, the 7-arc can cross over several arcs whose colors are 0, 3, 6, 7, 8 missing 4 similarly to the proof of Lemma 6.2. We remark that when the 7-arc crosses over an 0- or 3-arc, then we have a pair of new crossings of color $\{0\mid 7\mid 3\}$. We deform the diagram near every 3-arc of type (ii) with a 7-arc to remove all the 3-arcs of type (ii). See the center of the figure.

Now, since every crossing with a 3-arc is of color $\{0\mid 7\mid 3\}$, every 3-arc is of type (iii). We deform the diagram near every 3-arc of type (iii) with a 7-arc as shown in the right of the figure so that we obtain a diagram with no 3-arc. \square

Corollary 6.3. *For any 11-colorable knot K and $a \neq b \in \mathbb{Z}/11\mathbb{Z}$, there is an 11-colored diagram (D, C) of K with*

$$\text{Im}(C) = \{a, b, 5a + 7b, 2a + 10b, 10a + 2b\}.$$

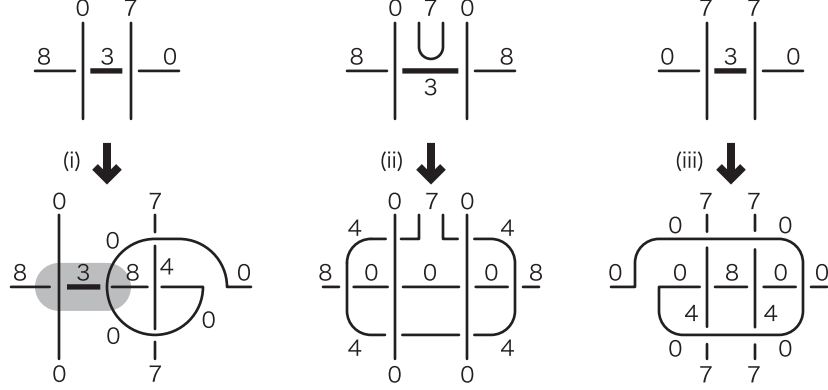


FIGURE 22.

Proof. Let $f : \mathbb{Z}/11\mathbb{Z} \rightarrow \mathbb{Z}/11\mathbb{Z}$ be the affine map defined by $f(x) = 3(b-a)x + a$. Since the map f satisfies

$$f(0) = a, \quad f(4) = b, \quad f(6) = 5a + 7b, \quad f(7) = 2a + 10b, \quad f(8) = 10a + 2b,$$

we have the conclusion by Lemma 2.10 and Theorem 1.1(ii). \square

7. 11-COLORABLE RIBBON 2-KNOT

A *ribbon 2-knot* [3] is a kind of knotted 2-sphere embedded in \mathbb{R}^4 . Such a 2-knot is presented by a diagram in \mathbb{R}^3 with only double point circles [18], the n -colorability is defined similarly to the classical case by assigning an element of $\mathbb{Z}/n\mathbb{Z}$ to each sheet of the diagram. Refer to [1] for a diagram of a knotted surfaces.

Lemma 7.1. *Let K be an 11-colorable ribbon 2-knot. For each set $S = \{1, 4, 6, 7, 8\}$ or $\{0, 4, 6, 7, 8\}$, there is an 11-colored diagram of K which satisfies the following.*

- (i) *Every double point circle has a neighborhood as shown in Figure 23, and all the sheets of the diagram other than the small shaded disks are colored by S .*
- (ii) *While the color $2a - b$ of the shaded disk may not belong to S , the pair (a, b) must satisfy $2b - a \in S$.*

Proof. Let A be a virtual arc which presents K [16]. Since K is 11-colorable, so is A . Then there is an 11-colored diagram (D, C) of A with $\text{Im}(C) = S$ by a similar argument in the proof of Theorems 1.1. The diagram of K associated to (D, C) is the desired one [13, 17]. \square

Theorem 7.2. *Any 11-colorable ribbon 2-knot satisfies the following.*

- (i) *There is an 11-colored diagram (D_1, C_1) of K with $\text{Im}(C_1) = \{1, 4, 6, 7, 8\}$.*
- (ii) *There is an 11-colored diagram (D_2, C_2) of K with $\text{Im}(C_2) = \{0, 4, 6, 7, 8\}$.*

Proof. (i) We may assume that (D, C) satisfies Lemma 7.1 for $S = \{1, 4, 6, 7, 8\}$. In the left of Figure 23, the shaded disk is colored $2b - a$. The pair (a, b) with $a, b, 2b - a \in S$ and $2a - b \notin S$ is one of the following:

$$(a, b) = (4, 1), (4, 7), (1, 7), (1, 6), (7, 6), (7, 8), (6, 8), (6, 4), (8, 4), (8, 1).$$

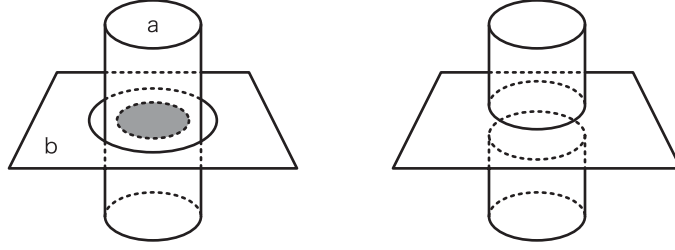


FIGURE 23.

In fact, each edge $\{x|y|z\}$ in the palette graph $G(S)$ produces such two pairs (y, x) and (y, z) .

First, we consider the case $(a, b) = (4, 1)$, where the shaded sheet is colored 9. There is an 8-sheet in (D, C) . We pull the 8-sheet toward the 9-sheet without introducing new double points and deform the diagram as shown in the left of Figure 24 to remove the 9-sheet. We remark that the figure shows a cross-section of the neighborhood of the 9-sheet. Next, we consider the case $(a, b) = (4, 7)$, where the shaded sheet is colored 10. We deform the horizontal 4-sheet by surrounding the 10-sheet, that reduces the case $(a, b) = (4, 1)$. See the right of the figure.

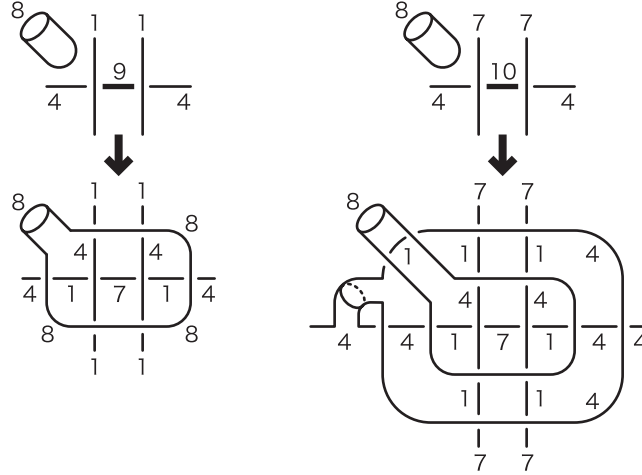


FIGURE 24.

Let $f : \mathbb{Z}/11\mathbb{Z} \rightarrow \mathbb{Z}/11\mathbb{Z}$ be the affine map defined by $f(x) = 9x + 9$. Since we have

$$f(1) = 7, f(4) = 1, f(6) = 8, f(7) = 6, \text{ and } f(8) = 4,$$

the cases $(a, b) = (1, 7)$, $(7, 6)$, $(6, 8)$, and $(8, 4)$ are obtained from $(a, b) = (4, 1)$ by applying f repeatedly, and the cases $(a, b) = (1, 6)$, $(7, 8)$, $(6, 4)$, and $(8, 1)$ are obtained from $(a, b) = (4, 7)$ similarly.

(ii) We may assume that (D, C) satisfies Lemma 7.1 for $S = \{0, 4, 6, 7, 8\}$. The pair (a, b) with $a, b, 2b - a \in S$ and $2a - b \notin S$ is one of the following:

$$(a, b) = (4, 8), (7, 6), (7, 8), (6, 8), (6, 4), \text{ and } (0, 7).$$

In fact, each edge $\{x|y|z\}$ in the palette graph $G(S)$ produces such two pairs (y, x) and (y, z) other than $(4, 8)$ from $\{0|4|8\}$ and $(0, 4)$ from $\{4|0|7\}$.

For the case $(a, b) = (4, 8)$, we deform the horizontal 4-sheet by surrounding the shaded 1-sheet as shown in the left of Figure 25 so that we can remove the 1-sheet. The case $(a, b) = (0, 7)$ can be similarly proved. See the right of the figure.

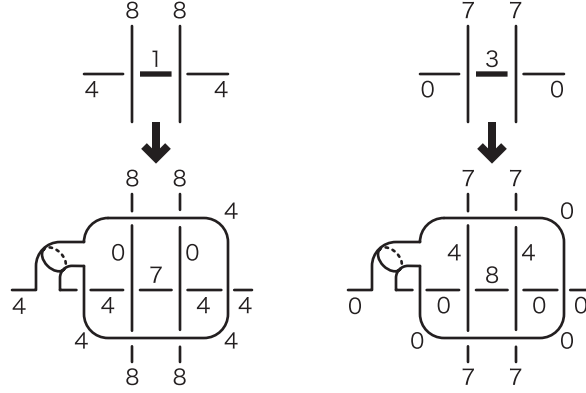


FIGURE 25.

For the case $(a, b) = (7, 6)$, we pull a 0-sheet and deform the diagram as shown in the left of Figure 26. Then we can remove the 5-sheet without introducing new colors. For the case $(a, b) = (7, 8)$, we first deform the horizontal 7-sheet by surrounding the shaded 9-sheet, that reduces to the case $(a, b) = (7, 6)$.

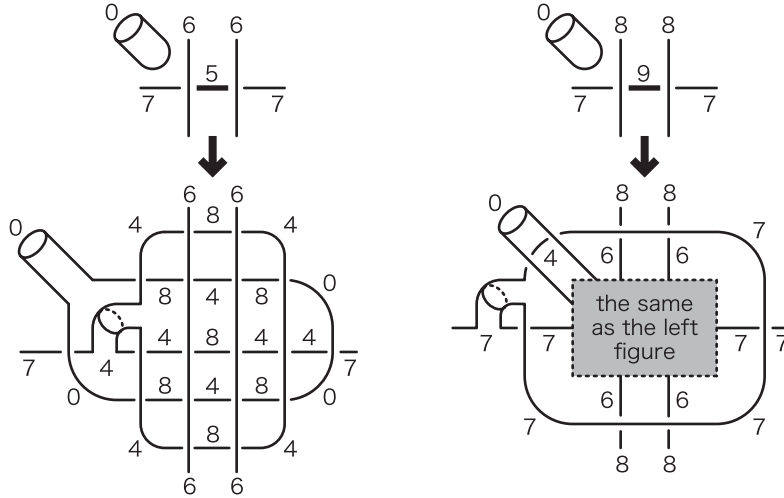


FIGURE 26.

For the case $(a, b) = (6, 8)$, we pull a 7-sheet and surround the shaded 10-sheet by the 7-sheet as shown in the left of Figure 27 so that the color 10 is removed.

For the case $(a, b) = (6, 4)$, we pull a 7-sheet toward the shaded 2-sheet and deform the horizontal 6-sheet to surround the 2-sheet. Then this case reduces to the case $(a, b) = (6, 8)$. \square

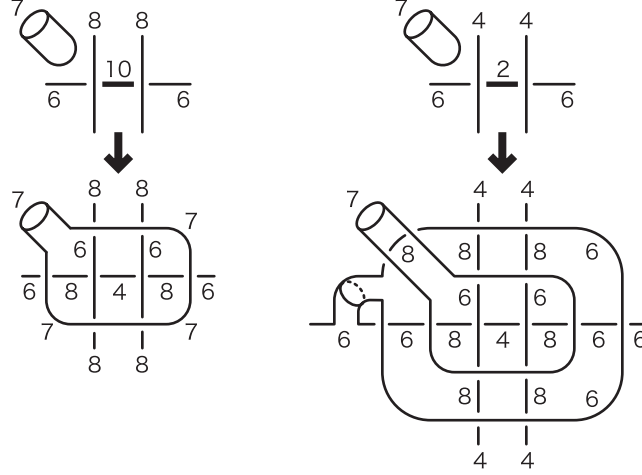


FIGURE 27.

For a p -colorable 2-knot K , we denote by $C_p(K)$ the minimum number of $\#\text{Im}(C)$ for all non-trivially p -colored diagrams (D, C) of K [17]. Then the following is an immediate consequence of Theorem 7.2.

Corollary 7.3. *Any 11-colorable ribbon 2-knot K satisfies $C_{11}(K) = 5$.* \square

The proof of the following is as same as that of Corollaries 3.2 and 6.3.

Corollary 7.4. *For any 11-colorable ribbon 2-knot K and $a \not\equiv b \in \mathbb{Z}/11\mathbb{Z}$, there are 11-colored diagrams (D_1, C_1) and (D_2, C_2) of K with*

$$\begin{aligned} \text{Im}(C_1) &= \{a, b, 3a + 9b, 10a + 2b, 6a + 6b\}, \text{ and} \\ \text{Im}(C_2) &= \{a, b, 5a + 7b, 2a + 10b, 10a + 2b\}. \end{aligned}$$

\square

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